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Journal of APPLIED MATHEMATICS AND MECHANICS

www.elsevier.com/locate/jappmathmech

Journal of Applied Mathematics and Mechanics 69 (2005) 222–233

A NON-STATIONARY ANALOGUE OF CHAPLYGIN'S EQUATIONS IN ONE-DIMENSIONAL GAS DYNAMICS†

A. I. RYLOV

Novosibirsk

e-mail: rylov@math.nsc.ru

(*Received* 16 *January* 2004)

As an extension of the results obtained in [1], two equivalent uniformly divergent systems of equations are constructed in the speedograph plane, each of which is the analogue of Chaplygin's equation in the hodograph plane. Each of the systems reduces to a linear second-order equation, in one case for the particle function (the Lagrange coordinate) ψ, and in the other for the time t. These systems possess an infinite set of exact solutions. It is shown that a uniformly divergent system of first-order equations correspond to each of these, and, related to them, the simplest non-linear homogeneous second-order equation in the modified events plane (ψ, t) and the conservation law in the events plane (x, t). Clear relations are obtained between the velocities of the fronts of constant values of the newly constructed dependent variables and the velocity of sound. Examples are given which demonstrate the relation between the exact solutions with the uniformly divergent equations and the conservation laws of one-dimensional non-stationary gas dynamics and, simultaneously, enable one to compare the newly obtained results (the exact solutions, the equations and conservation laws, and the relations for the velocities of the front) with existing results, including those for plane steady flows. The so-called additional conservation laws, to which Godunov drew attention, are considered.

1. THE NON-STATIONARY ANALOGUE OF CHAPLYGIN'S EQUATIONS

Consider the one-dimensional non-stationary barotropic flows of an ideal (non-viscous and non-heat conducting) gas with plane waves. We will take as the initial system of equations the system of equations in the plane of events (x, t) [2–4]

$$\rho u_x + u\rho_x + \rho_t = 0, \quad \rho u u_x + a^2\rho_x + \rho u_t = 0 \tag{1.1}$$

Here t is the time, x is the geometrical coordinate, u is the velocity, ρ is the density, p is the pressure and a is the velocity of sound $\left(a^2 = \dfrac{dp}{d\rho}\right)$.

After changing to the speedograph plane (u, υ), the system (1.1) can be rewritten as [2]

$$at_\upsilon - ut_u + x_u = 0, \quad ut_\upsilon - at_\upsilon - x_\upsilon = 0 \tag{1.2}$$

The function υ, introduced previously by Reimann, is defined by the following equivalent equations [2, 5].

$$\upsilon = \int\frac{a}{\rho}d\rho, \quad \frac{d\upsilon}{d\rho} = \frac{a}{\rho}$$

In view of the fact that the flow is barotropic, any three of the four functions ρ, p, a and υ can be expressed in terms of the remaining fourth function. In system (1.2) we used υ as this function. For our further investigation in turns out to be more suitable to use the functions p or $w = \rho^{-1}$ [3, 5].

†*Prikl. Mat. Mekh.* Vol. 69, No. 2, pp. 245–257, 2005.

0021–8928/$—see front matter.

doi: 10.1016/j.jappmathmech.2005.03.008

For greater clarity, we will confine ourselves to the case of a polytropic gas with adiabatic exponent k; we have

$$p = \rho^k, \quad a^2 = \frac{dp}{d\rho} = k\frac{p}{\rho} = k\rho^{k-1}, \quad \upsilon = \frac{2a}{k-1}$$

At the fourth step for converting the homogeneous linear system (1.2) into a uniformly divergent system, we will consider, instead of the x coordinate, the particle function (the Lagrange coordinate) ψ, defined in the standard way [2–5]

$$\frac{\partial\psi}{\partial x} = \rho, \quad \frac{\partial\psi}{\partial t} = -\rho u, \quad d\psi = \rho dx - \rho u dt$$

As a result, we can rewrite system (1.2) as

$$\rho a t_\upsilon + \psi_u = 0, \quad \rho a t_u + \psi_\upsilon = 0$$

We can achieve a further simplification by replacing the function υ by p or $w = \rho^{-1}$. As a result, we obtain in the first case a linear uniformly divergent system

$$\rho^2 a^2 \frac{\partial t}{\partial p} + \frac{\partial\psi}{\partial u} = 0, \quad \frac{\partial t}{\partial u} + \frac{\partial\psi}{\partial p} = 0 \tag{1.3}$$

and its equivalent linear homogeneous second-order equation for the function ψ.

$$\frac{\partial^2\psi}{\partial u^2} - \rho^2 a^2 \frac{\partial^2\psi}{\partial p^2} = 0 \tag{1.4}$$

In the second case we have the system

$$\rho^2 a^2 \frac{\partial t}{\partial u} - \frac{\partial\psi}{\partial w} = 0, \quad \frac{\partial t}{\partial w} - \frac{\partial\psi}{\partial u} = 0 \tag{1.5}$$

and the second-order equation for the function t

$$\frac{\partial^2 t}{\partial w^2} - \rho^2 a^2 \frac{\partial^2 t}{\partial u^2} = 0 \tag{1.6}$$

The equations in systems (1.3) and (1.5) only contain two terms. In these systems and in the second-order equations (1.4) and (1.6) there is only one coefficient $\rho^2 a^2$, which differs from ± 1, which, in turn, depends on only one independent variable, p or w. Hence, each of the hyperbolic systems (1.3) and (1.5) can be considered as an analogue of the Chaplygin system of equations in the supersonic part of the hodograph plane [2–4, 6]. Similarly, each of the second-order hyperbolic equations (1.4) and (1.6) can be regarded as the analogue of second-order Chaplygin equations [2–4].

Systems (1.3) and (1.5) and Eqs (1.4) and (1.6) have an infinite set of exact solutions. Since systems (1.3) and (1.5) are equivalent, henceforth, to fix our ideas, we will confine ourselves to considering the following solutions of system (1.3)

$$t = f(p, u), \quad \psi = g(p, u) \tag{1.7}$$

An infinite set of such solutions can be constructed by the method of separation of variables. For the polytropic gases considered, the coefficient $\rho^2 a^2$ is a power function of the independent variable p, which reduces the process of constructing a solution to a well-known problem [7, 8]. Moreover, one should note the finite number of obvious solutions, which are linear functions of p (or w) and u, their products and squares and certain quadratures. Examples of such solutions will be given in Section 3.

Note that exact solutions are also known for other forms of writing the equations of one-dimensional non-stationary gas dynamics [2–5, 6]. The role of the exact solutions in constructing and analysing flows is well known. At the same time, it turns out that the structure of Eqs (1.3) and the exact solutions of the form (1.7) play an important role when constructing uniformly divergent equations in the (ψ, t) plane, as a consequence, new conservation laws in the (ψ, t) and (x, t) planes. The following section is devoted to these problems.

2. EXACT SOLUTIONS IN THE SPEEDOGRAPH PLANE AND THE CORRESPONDING EQUATIONS IN THE EVENTS PLANE

We will consider the relation between the structure of system (1.3), its solutions (1.7) and new forms of the equations of one-dimensional non-stationary gas dynamics in the (ψ, t) plane. In this connection, we first recall [9] that if an arbitrary homogeneous or non-homogeneous first-order system with independent variables p and u and dependent variables t and ψ has the two-parameter solution

$$\alpha(t, \psi, p, u) = a, \quad \beta(t, \psi, p, u) = b \tag{2.1}$$

with arbitrary constants a and b, then the introduction of new dependent variables $U = \alpha(t, \psi, p, u)$ and $V = \beta(t, \psi, p, u)$ converts the initial system into a new homogeneous system. Naturally, the presence in system (1.3) of an unbounded set of two-parameter solutions raises the question of using them to construct new homogeneous systems in the speedograph plane. Starting from this, using arbitrary constants a, b and c, we will convert solution (1.7) into its equivalent solution, but written in the form (2.1)

$$ct - f(p, u) = a, \quad c\psi - g(p, u) = b \tag{2.2}$$

after which we arrive at the following homogeneous system

$$\rho^2 a^2 (ct - f)_p + (c\psi - g)_u = 0, \quad (ct - f)_u + (c\psi - g)_p = 0 \tag{2.3}$$

System (2.3) can obviously be constructed without employing the results from [9] presented above, since it is a direct consequence of Eqs (1.3) and of the solutions (1.7). By itself it gives no new information on the solutions of the equations in the speedograph plane. At the same time, as will be shown below, on changing to the (ψ, t) plane, system (2.3) is of definite interest.

Theorem 1. For each non-degenerate solution (1.7) of system (1.3), the introduction of the dependent variables

$$U = f(p, u), \quad V = g(p, u), \quad (J = f_p g_u - f_u g_p \neq 0)$$

and the functions R, defined in terms of the derivatives

$$R_t = g(p, u), \quad R_\psi = f(p, u)$$

there is a corresponding system of uniformly divergent equations of one-dimensional non-stationary gas dynamics in the (ψ, t) plane

$$\rho^2 a^2 U_\psi - V_t = 0, \quad U_t - V_\psi = 0 \tag{2.4}$$

a second-order equation

$$\rho^2 a^2 R_{\psi\psi} - R_{tt} = 0 \tag{2.5}$$

and a conservation law in the (x, t) plane

$$(\rho u U - V)_x + (\rho U)_t = (\rho u f - g)_x + (\rho f)_t = 0 \tag{2.6}$$

Proof. From the fact that relations (1.7) are a solution of system (1.3), we have the equalities

$$\rho^2 a^2 U_p + V_u = \rho^2 a^2 (U_t t_p + U_\psi \psi_p) + V_t t_u + V_\psi \psi_u = 0$$

$$U_u + V_p = U_t t_u + U_\psi \psi_u + V_t t_p + V_\psi \psi_p = 0$$

Expressing the derivative of ψ_p and ψ_u, using system (1.3), in terms of the derivatives of t_p and t_u, we obtain a homogeneous linear algebraic system for the quantities $X = U_t - V_\psi$ and $Y = \rho^2 a^2 U_\psi - V_t$

$$\rho^2 a^2 t_p X - t_u Y = 0, \quad t_u X - t_p Y = 0 \tag{2.7}$$

From the condition of the theorem, the Jacobian is as follows:

$$J = f_p g_u - f_u g_p = t_p \psi_u - t_u \psi_p = -\rho^2 a^2 t_p^2 + t_u^2 \neq 0$$

Consequently, system (2.7) has the unique solution $X = 0$, $Y = 0$, which also leads to Eqs (2.4) and, as a consequence, to the second-order equation (2.5).

To construct conservation law (2.6), we will rewrite the second (divergent) equality of system (2.4) using the independent variables (x, t). We obtain

$$uU_x + U_t - \rho^{-1} V_x = 0$$

Adding this equality to the obviously identity

$$\rho^{-1} U((\rho u)_x + \rho_t) = 0$$

we arrive at Eq. (2.6), which completes the proof of the theorem.

Corollary. System (2.4) allows of the following generalization

$$\rho^2 a^2 (U - ct)_\psi - (V - c\psi)_t = 0, \quad (U - ct)_t - (V - c\psi)_\psi = 0$$

Hence c is an arbitrary constant.

It can be seen that on changing to the speedograph plane this system is identical with system (2.3) considered above.

Remarks 1. Equation (2.6) gives an unbounded set of conservation laws in the (x, t) plane. The finite number of conservation laws of gas dynamics in (x, y, z, t) space was constructed earlier in [10–12]. Some of these retain their meaning for the (x, t) plane also. It should be emphasised that on page 19 in [12], which was prepared for press and published later than [11], it is stated that the addition to the system of equations of gas dynamics investigated in [11, 12], of new relations, which do not contradict it, may extend the system of conservation laws constructed in [11, 12]. These relations include the potentiality condition, relations which occur when the number of variables is reduced, etc. Results of this paper and of the paper which preceded it [1] clearly demonstrate certain possible ways of realizing the assertion from [12] stated above.

2. For individual exact solutions, the function R introduced above has a clear physical meaning, which will also be demonstrated by a number of examples.

Each of systems (2.4) is in many ways identical to the system of equations of two-dimensional steady flows in the potential plane [3] and each of the infinite set of systems obtained using the solutions of Chaplygin's equations [1]. Hence, we might expect that the structure of the level lines $U = \text{const}$ and $V = \text{const}$ in the (x, t) plane also has much in common with the structure of the level lines of steady flows [13]. We have the following theorem.

Theorem 2. For the velocities U^c and V^c of the fronts $U = \text{const}$ and $V = \text{const}$, corresponding to the solution of system (2.4), the following relations, connecting them with the velocities of the gas u and the velocity of sound a and with the Jacobian $j = U_\psi V_t - U_t V_\psi$ of system (2.4) hold

$$(U^c - u)(V^c - u) = a^2 \tag{2.8}$$

$$\rho^2 U_\psi^2 (a^2 - (U^c - u)^2) = a^{-2} V_\psi^2 ((V^c - u)^2 - a^2) = j \tag{2.9}$$

In other words, the relative velocities of the fronts $(U^c - u)$ and $(V^c - u)$ have the same sign, their geometric mean velocity is equal to the velocity of sound, and the equations $U^c - u = V^c - u = a$ are only possible when $j = 0$, and, finally, depending on the sign of the Jacobian j the following inequalities are satisfied

$$j > 0, \quad (U^c - u)^2 < a^2 < (V^c - u)^2$$

$$j < 0, \quad (V^c - u)^2 < a^2 < (U^c - u)^2$$

Proof. Along the line U = const we have

$$U_\psi d\psi + U_t dt = U_\psi(\rho dx - \rho u dt) + U_t dt = U_\psi \rho(U^c - u)dt + U_t dt = 0$$

For the line V = const all the relations can be written similarly. Consequently,

$$U^c - u = -\frac{U_t}{\rho U_\psi}, \quad V^c - u = -\frac{V_t}{\rho V_\psi} \tag{2.10}$$

whence, taking system (2.4) into account, we also obtain the required equality (2.8). Further, the expression for the Jacobian $j = U_\psi V_t - U_t V_\psi$ using equalities (2.4) and (2.10) leads to relations (2.9), which completes the proof.

Remarks. 1. The class of flows of an incompressible fluid, covered by shallow-water theory [3, 6], enables us to observe visually the changes in one of the functions being investigated and the motion of the front of its constant values. These flows are described by the equations of one-dimensional non-stationary gas dynamics (2.4), in which we take

$$U = -u, \quad V = \frac{gh^2}{2}, \quad p = \frac{gh^2}{2}, \quad \rho = h, \quad a^2 = gh$$

where g is the acceleration due to gravity, $h = h\,(x, t)$ is the distance from the free surface to the horizontal bottom and ψ, x and t are connected by the relation $d\psi = hdx - hudt$. The change in the value of the h and the velocity V^c of the displacement of a point on the free surface h = const can be easily observed and measured experimentally. Substituting the values of h and V^c into Eq. (2.8), in which a^2 is replaced by gh, we obtain a relation connecting the velocity of the flow u and the velocity V^c of motion of the front u = const.

2. Theorem 2 is largely identical to the corresponding theorem in [13] on the structure of the level lines of plane steady flows, for which an infinite set of uniformly divergent equations has also been constructed [1]. In the supersonic case, this theorem can be formulated as follows:

The geometric mean of the tangents of the angles formed by the level lines of the functions investigated with the velocity vector is equal to the tangent of the Mach angle.

3. The inequalities presented in Theorem 2, which follow from relations (2.9), do not contradict the fact that, for a gas with constant parameters, a perturbation propagates with the velocity of sound. In this case, as is well known, the simple wave, in which the Jacobian $j = 0$, adjoins the constant solution, by virtue of which, at each point of the simple wave, including on the boundary with the constant solution, $U^c - u = V^c - u = a$. At the same time, at each point of the region of the solution of general form c with non-zero Jacobian j, depending on its sign, either $|U^c - a| > a$ or $|V^c - u| > a$, i.e. one of the fronts, either U = const or V = const, may move more rapidly than a sound wave.

3. EXAMPLES

We will consider some examples of the solutions of the form $t = f$ and $\psi = g$ of systems (1.3) and (1.5) and the uniformly divergent systems (2.4) and the second-order equations (2.5) in the (ψ, t) plane and the conservation laws (2.6) in the (x, t) plane related to them, according to Theorem 1. The construction of these solutions does not require the use of separation of the variables. Nevertheless, these examples also clearly demonstrate the relation between the exact solutions in the speedograph plane and the first- and second-order equations and the conservation laws in the event planes (ψ, t) and (x, t). Section 4 is devoted to solutions, the construction of which requires the use of separation of the variables. It should also be noted that some of the solutions of this and subsequent sections are of independent interest.

Example 1

$$t = f_1 = -u, \quad \psi = g_1 = p, \quad R(\psi, t) = P = \int_{t_0}^{t} p(\psi, t)d\tau \tag{3.1}$$

$$-\rho^2 a^2 u_\psi - p_t = 0, \quad -u_t - p_\psi = 0 \tag{3.2}$$

$$\rho^2 a^2 P_{\psi\psi} - P_{tt} = 0 \tag{3.3}$$

$$-(p+\rho u^2)_x-(\rho u)_t = 0 \tag{3.4}$$

The integral occurring here is an integral functional. It associates with the section (t_0, t) of trajectory ψ = const a number p, which is connected directly with the law of conservation of momentum (the second of Eqs (3.2) and Eq. (3.4)). In particular, the quantity P can also be written in terms of the integral over the Lagrange coordinate for a fixed value of the time, which can be seen from the following form of writing the law of conservation of momentum

$$P(\psi, t)-P(\psi_0, t) = -\int_{\psi_0}^{\psi} u(\xi, t)d\xi+\int_{\psi_0}^{\psi} u(\xi, t_0)d\xi$$

Equations (3.2) and the law of conservation of momentum (3.4) are well known [2–5, 10–12]. The second-order equation (3.3) may be of some interest.

Example 2

$$t = f_2 = w = \rho^{-1}, \quad \psi = g_2 = u, \quad R(\psi, t) = x \tag{3.5}$$

$$\rho^2a^2w_\psi-u_t = 0, \quad w_t-u_\psi = 0 \tag{3.6}$$

$$\rho^2a^2x_{\psi\psi}-x_{tt} = 0 \tag{3.7}$$

$$(\rho uf_2-g_2)_x+(\rho f_2)_t = 0_x+1_t = 0 \tag{3.8}$$

As can be seen, the conservation law in the (x, t) and (ψ, t) planes reflects the obvious fact that, when moving over a closed contour, the increment of the x coordinate is equal to zero.

Equations (3.6) are well known [2–5]. The example considered is a more interesting extremely simple form of the function $R = x$, which leads to a compact form of the second-order equation (3.7), in which the coefficient ρ^2a^2 is expressed in terms of the derivative x_ψ. This equation was constructed earlier in [4] by another method.

The equality of the function R to the geometrical coordinate x provides an appropriate comparison with similar equations for plane steady flows. This comparison is interesting in that plane flows are characterized by the presence of two geometrical coordinates, x and y. In this connection, we recall two equally justified systems of equations, constructed [1] using Ringleb's solutions [2]

$$\frac{1-M^2}{\rho^2}\left(\frac{\sin\theta}{q}\right)_\varphi+\left(\frac{\cos\theta}{\rho q}\right)_\psi = 0, \quad \left(\frac{\sin\theta}{q}\right)_\psi-\left(\frac{\cos\theta}{\rho q}\right)_\varphi = 0$$

$$\frac{1-M^2}{\rho^2}\left(\frac{\cos\theta}{q}\right)_\varphi-\left(\frac{\sin\theta}{\rho q}\right)_\psi = 0, \quad \left(\frac{\cos\theta}{q}\right)_\psi+\left(\frac{\sin\theta}{\rho q}\right)_\varphi = 0$$

In each of the systems, the dependent variables are the expressions in parenthesis, and in this subsection we will use standard notation for plane steady flows: M is the Mach number, q and θ are the modulus and angle of inclination of the velocity vector, φ is the potential and ψ is the stream function.

Following the well-known approach [4], using the results of this paper and noting that

$$x_\varphi = \frac{\cos\theta}{q}, \quad x_\psi = -\frac{\sin\theta}{\rho q}, \quad y_\varphi = \frac{\sin\theta}{q}, \quad y_\psi = \frac{\cos\theta}{\rho q}$$

we arrive at two new and equally justified second-order equations for plane potential flows

$$\frac{1-M^2}{\rho^2}y_{\varphi\varphi}+y_{\psi\psi} = 0, \quad \frac{1-M^2}{\rho^2}x_{\varphi\varphi}+x_{\psi\psi} = 0$$

each of which can be regarded as the inverse of the well-known second-order equation for the potential in the (x, y) plane [2, 4–6].

Example 3

$$t = f_3 = -\frac{1}{2}u^2 - \frac{k-1}{4k}\upsilon^2, \quad \psi = g_3 = up$$
$$R(\psi, t) = W = \int_{t_0}^{t} p(\psi, \tau)u(\psi, \tau)d\tau \tag{3.9}$$

$$-\rho^2 a^2\left(\frac{u^2}{2} + \frac{k-1}{4k}\upsilon^2\right)_\psi - (pu)_t = 0, \quad -\left(\frac{u^2}{2} + \frac{k-1}{4k}\upsilon^2\right)_t - (pu)_\psi = 0 \tag{3.10}$$

$$\rho^2 a^2 W_{\psi\psi} - W_{tt} = 0 \tag{3.11}$$

$$-\left(\rho u\left(\frac{u^2}{2} + \frac{k-1}{4k}\upsilon^2\right) + pu\right)_x - \left(\rho\left(\frac{u^2}{2} + \frac{k-1}{4k}\upsilon^2\right)\right)_t = 0 \tag{3.12}$$

The first relation of (3.9), dividing both its sides by t, can be written in the form of the equation of an ellipse in the speedograph plane (u, υ).

The functional W has a clear physical meaning: it is the work performed by the trajectory (piston) with a value of the Lagrange coordinate ψ at the instant of time t (the integral which defines the function W is evaluated along the trajectory ψ = const). Solution (3.9) and system (3.10) corresponding to it as well as the second-order equation (3.11) are fairly new, whereas the conservation law (3.12) in the (x, t) plane in the (ψ, t) plane (the second of Eqs (3.10)) is the well-known law of conservation of energy [2–5, 10–12]. Using this case, we note that this law of conservation enables us to reduce the fairly complex problem of constructing the optimum motion of a piston (optimum from the point of view of achieving maximum work) to a one-dimensional variational problem for the closing characteristic and enables us to obtain the conditions of optimality in the form of finite algebraic relations [14].

Example 4

$$t = f_4 = \frac{u}{\rho}, \quad \psi = g_4 = \frac{u^2}{2} - \frac{a^2}{k-1}, \quad R = \varphi(\psi, t) \tag{3.13}$$

$$\rho^2 a^2\left(\frac{u}{\rho}\right)_\psi - \left(\frac{u^2}{2} - \frac{a^2}{k-1}\right)_t = 0, \quad \left(\frac{u}{\rho}\right)_t - \left(\frac{u^2}{2} - \frac{a^2}{k-1}\right)_\psi = 0 \tag{3.14}$$

$$\rho^2 a^2 \frac{\partial^2\varphi(\psi, t)}{\partial\psi^2} - \frac{\partial^2\varphi(\psi, t)}{\partial t^2} = 0 \tag{3.15}$$

$$\left(\frac{u^2}{2} + \frac{a^2}{k-1}\right)_x + u_t = 0 \tag{3.16}$$

Here φ is the flow potential, defined in the standard way: $\frac{\partial\varphi(x, t)}{\partial x} = u$. For the potential φ, the Cauchy–Lagrange integral derived below, in particular, and the second-order equation [2–4,6]

$$\frac{\partial\varphi(x, t)}{\partial t} + \frac{u^2}{2} + \frac{a^2}{k-1} = 0 \tag{3.17}$$

$$\frac{\partial^2\varphi(x, t)}{\partial t^2} + 2u\frac{\partial^2\varphi(x, t)}{\partial x\partial t} + (u^2 - a^2)\frac{\partial^2\varphi(x, t)}{\partial x^2} = 0 \tag{3.18}$$

are well known.

Example 4 thereby demonstrates the relation between the exact solutions (3.13) not only with the new forms of the equations, but also with the potential φ, and also takes into account the fact that when the independent variables (x, t) are replaced by (ψ, t) the second-order equation (3.18) becomes Eq. (3.15) and is considerably simplified.

The choice of Examples 1–4 is fairly random, and is only related to the presence of the simplest solutions $t = f_i(p, u)$ and $\psi = g_i(p, u)$, which is the basis of these examples. Nevertheless, these examples are also interesting from the procedural point of view, since they demonstrate the presence of a linear relation between the derivatives $(f_i)_t$ and $(g_i)_t$, and as a consequence, the divergent equations from Examples 1–4.

For examples 1–3 we have

$$-u(f_1)_t - p(f_2)_t + (f_3)_t = uu_t - p\left(\frac{1}{\rho}\right)_t - \left(\frac{u^2}{2} + \frac{p}{\rho(k-1)}\right)_t = 0$$

$$-u(g_1)_\psi - p(g_2)_\psi + (g_3)_\psi = -up_\psi - pu_\psi + (pu)_\psi = 0$$

after which, we obtain a similar linear relation connecting the divergent equations from Examples 1–3:

$$-u((f_1)_t - (g_1)_\psi) - p((f_2)_t - (g_2)_\psi) + ((f_3)_t - (g_3)_\psi) = u(u_t + p_\psi) - p\left(\left(\frac{1}{\rho}\right)_t - u_\psi\right) -$$

$$-\left(\left(\frac{u^2}{2} + \frac{p}{\rho(k-1)}\right)_t + (pu)_\psi\right) = 0 \qquad (3.19)$$

Consequently, according to what was said earlier in [15, 16], each of the three conservation laws considered is additional to the two other conservation laws

Similar linear relations also exist for Examples 1, 2 and 4

$$-\frac{1}{\rho}(f_1)_t + u(f_2)_t - (f_4)_t = \frac{1}{\rho}u_t + u\left(\frac{1}{\rho}\right)_t - \left(\frac{u}{\rho}\right)_t = 0 \qquad (3.20)$$

$$-\frac{1}{\rho}(g_1)_\psi + u(g_2)_\psi - (g_4)_\psi = -\frac{1}{\rho}p_\psi + uu_\psi + \left(\frac{u^2}{2} - \frac{p}{\rho}\frac{k}{(k-1)}\right)_\psi = 0 \qquad (3.21)$$

$$-\frac{1}{\rho}((f_1)_t - (g_1)_\psi) + u((f_2)_t - (g_2)_\psi) - ((f_4)_t - (g_4)_\psi) =$$

$$= \frac{1}{\rho}(u_t + p_\psi) + u\left(\left(\frac{1}{\rho}\right)_t - u_\psi\right) - \left(\left(\frac{u}{\rho}\right)_t - \left(\frac{u^2}{2} - \frac{p}{\rho}\frac{k}{(k-1)}\right)_\psi\right) = 0 \qquad (3.22)$$

It is also easy to see that the system of equations (3.19) and (3.22) enable one to express the divergent equations from Examples 1 and 2 in terms of the divergent equations from Examples 3 and 4. One can thereby take as proved the fact that each of the divergent equations from Examples 1–4 can be written in the form of a linear combination of two other divergent equations from the same examples. Consequently, according to what was stated earlier in [15, 16], each of the conservation laws from Examples 1–4 is additional to the other conservation laws form these examples.

It should be noted that this is a fairly typical situation, when a linear combination of three or more conservation laws, written in differential form, vanishes. Thus, in the papers mentioned above [15, 16], the construction of an additional conservation law is used to symmetrize the equations of gas dynamics. The problem of additional conservation laws was not discussed in [11, 12], but a simple analysis of the divergent equations derived in [11, 12] enables us to indicate several linear combinations which vanish. Here we must emphasise that the fact that these and other linear combinations mentioned above vanish can in no way be transferred to integral forms of the corresponding conservation laws. All this indicates that the classification of the conservation laws is fairly complex and far from complete. Note that this classification must take into account many factors, for example, whether the divergent equation of the composite part of the homogeneous system of equations of gas dynamics considered depends on the functions A and B from the equation $A_t - B_\psi = 0$ or on ψ, t, p and u or not, or whether it is possible that they also depend on the integral functions, as, for example, P, W, x and φ. As indicated, a more detailed consideration of examples 1–4 confirms this.

Consider examples 1 and 2. To do this we will multiply the second-order equations (3.3) and (3.7) by x and P respectively. Adding the results, we obtain the second-order inhomogeneous equation

$$\rho^2 a^2 (xP)_{\psi\psi} - (xP)_{tt} + 2\rho a^2 u + 2up = 0$$

and the corresponding non-uniformly divergent system

$$\rho^2 a^2 \left(\frac{P}{\rho} - xu\right)_{\psi} - (uP + xp)_t + 2\rho a^2 u + 2up = 0 \tag{3.23}$$

$$((xP)_{\psi})_t - ((xP)_t)_{\psi} \equiv \left(\frac{P}{\rho} - xu\right)_t - (uP + xp)_{\psi} = 0 \tag{3.24}$$

Divergent equation (3.24), which is of interest, gives a new consideration law in the (ψ, t) plane, which can also be written for the (x, t) plane.

We emphasize that the expressions in brackets in Eq. (3.24) include the integral functionals P and x. This fact does not complicate the use of the conservation law in integral form. It is only necessary, when evaluating the integral over a closed contour

$$\oint \left(\frac{P}{\rho} - xu\right) d\psi + (uP + xp) dt$$

to calculate simultaneously the quantities P and x using the integrals from (3.1) and (3.5), calculated over the same contour. At the same time, the use of the integral conservation laws, which follow from Eq. (3.24) and its similar additional conservation laws in variational problems on the optimum motion of a piston, for example, as an extension of the results obtained earlier [14], leads to the under-investigated situation when the minimized integral functional contains other integral functionals as the arguments.

By analogy with Eq. (3.24), we will write out the five remaining divergent equations

$$((PW)_{\psi})_t - ((PW)_t)_{\psi} \equiv -\left(uW + P\left(\frac{u^2}{2} + \frac{a^2}{k(k-1)}\right)\right)_t - (pW + Ppu)_{\psi} = 0$$

$$((P\varphi)_{\psi})_t - ((P\varphi)_{\psi})_t \equiv \left(-u\varphi + P\frac{u}{\rho}\right)_t - \left(p\varphi + P\left(\frac{u^2}{2} - \frac{a^2}{k-1}\right)\right)_{\psi} = 0$$

$$((xW)_{\psi})_t - ((xW)_t)_{\psi} \equiv \left(\frac{W}{\rho} - x\left(\frac{u^2}{2} + \frac{a^2}{k(k-1)}\right)\right)_t - (uW + xpu)_{\psi} = 0$$

$$((x\varphi)_{\psi})_t - ((x\varphi)_t)_{\psi} \equiv \left(\frac{\varphi}{\rho} + x\frac{u}{\rho}\right)_t - \left(u\varphi + x\left(\frac{u^2}{2} - \frac{a^2}{k-1}\right)\right)_{\psi} = 0$$

$$((\varphi W)_{\psi})_t - ((\varphi W)_t)_{\psi} \equiv \left(W\frac{u}{\rho} - \varphi\left(\frac{u^2}{2} + \frac{a^2}{k(k-1)}\right)\right)_t - \left(W\left(\frac{u^2}{2} - \frac{a^2}{k-1}\right) + \varphi pu\right)_{\psi} = 0$$

4. SOLUTIONS OBTAINED BY THE METHOD OF SEPARATION OF VARIABLES

Using the standard method, we write ψ in the form of the product $\psi = h(p)q(u)$, and after substituting this into Eq. (1.4), we have

$$\rho^2 a^2 \frac{h''(p)}{h(p)} = \frac{q''(u)}{q(u)} = A = \text{const} \tag{4.1}$$

The cases $A > 0$ and $A < 0$ require separate consideration.

A negative constant of separation, $A = -\lambda^2 < 0$. We have $q'' = -\lambda^2 q$. This leads to two independent solutions $q_1 = \cos\lambda u$ and $q_2 = \sin\lambda u$. Further, the second-order equation

$$\rho^2 a^2 h'' = kp^{1+1/k} h'' = Ah$$

also has two independent solutions, which, when $A = -\lambda^2$ we will denote by $S_1(p, \lambda)$ and $S_2(p, \lambda)$, and when $A = \lambda^2$ we will denote by $Z_1(p, \lambda)$ and $Z_2(p, \lambda)$. Fairly lengthy expressions for $S_i(p, \lambda)$ and $Z_i(p, \lambda)$ were derived previously in [7, 8]. Omitting the similar calculations, which relate to constructing a solution for t, we obtain, to simplify the writing, in versions 1 and 2, four collected pairs of functions, which give a solution of system (1.3) when $A = -\lambda^2$, and related to them, according to Theorem 1, uniformly divergent first-order systems and second-order equations in the (ψ, t) plane and conservation laws in the (x, t) plane. For version 1 we have

$$t = f = -\frac{S_i'}{\lambda}\sin\lambda u, \quad \psi = g = S_i\cos\lambda u$$

$$R_i^+(\psi, t) = \int_{t_0}^{t} S_i(\psi, \tau)\cos\lambda u(\psi, \tau)d\tau, \quad i = 1, 2$$

$$\rho^2 a^2 (R_i^+)_{\psi\psi} - (R_i^+)_{tt} \equiv -\rho^2 a^2 \left(\frac{S_i'}{\lambda}\sin\lambda u\right)_\psi - (S_i\cos\lambda u)_t = 0$$

$$((R_i^+)_\psi)_t - ((R_i^+)_t)_\psi \equiv -\left(\frac{S_i'}{\lambda}\sin\lambda u\right)_t - (S_i\cos\lambda u)_\psi = 0$$

$$\left(\rho u\frac{S_i'}{\lambda}\sin\lambda u + S_i\cos\lambda u\right)_x + \left(\rho\frac{S_i'}{\lambda}\sin\lambda u\right)_t = 0$$

(the prime on S_i and Z_i denotes differentiation with respect to p). Version 2 differs from version 1 by the fact that $\sin\lambda u$ has been replaced by $-\cos\lambda u$, and $\cos\lambda u$ has been replaced by $\sin\lambda u$, and also R_i^+ has been replaced by R_i^-.

By analogy with Theorem 2 from [1] it is easy to show that the two additional conservation laws correspond to the four conservation laws derived above. This is achieved by adding the divergent equations with subscript i from both versions, multiplied respectively by R_j^+ and R_j^-, where $j = 3 - i$, $i = 1, 2$. As a result we obtain the required additional conservation laws

$$((R_i^+)_\psi R_j^+ + (R_i^-)_\psi R_j^-)_t - ((R_i^+)_t R_j^+ + (R_i^-)_t R_j^-)_\psi \equiv$$
$$\equiv \left(-\frac{S_i'}{\lambda}\sin\lambda u R_j^+ + \frac{S_i'}{\lambda}\cos\lambda u R_j^-\right)_t - (S_i\cos\lambda u R_j^+ + S_i\sin\lambda u R_j^-)_\psi = 0$$

These conservation laws have much in common with the additional conservation laws derived at the end of the previous section. Thus, the derived divergent equations are also a composite part of the non-uniformly divergent equations of one-dimensional non-stationary gas dynamics.

A positive constant of separation, $A = \lambda^2 > 0$. It follows from relation (4.1) that $q'' = \lambda^2 q$, which leads to two independent solutions: $q_1 = e^{\lambda u}$ and $q_2 = e^{-\lambda u}$. Further, a noted above, the second-order equation for $h(p)$ also has two independent solutions, for which, when $A = \lambda^2 > 0$, the notation $Z_1 = Z_1(p, \lambda)$ and $Z_2 = Z_2(p, \lambda)$ is used. Omitting the similar calculations, which relate to constructing a solution for t, we obtain four pairs of functions, which give the solution of system (1.3) when $A = \lambda^2$ and the corresponding non-uniformly divergent first- and second-order equations in the (ψ, t) plane and the conservation laws in the (x, t) plane. We have

$$t = f = \mp\frac{Z_i'}{\lambda}e^{\pm\lambda u}, \quad \psi = g = Z_i e^{\pm\lambda u}$$

$$R_i^{\pm} = \int_{t_0}^{t} Z_i(\psi, \tau) e^{\pm\lambda u(\psi, \tau)} d\tau, \quad i = 1, 2$$

$$\rho^2 a^2 (R_i^{\pm})_{\psi\psi} - (R_i^{\pm})_{tt} \equiv \mp\rho^2 a^2 \left(\frac{Z_i'}{\lambda} e^{\pm\lambda u}\right)_{\psi} - (Z_i e^{\lambda u})_t = 0$$

$$((R_i^{\pm})_{\psi})_t - ((R_i^{\pm})_t)_{\psi} \equiv \mp\left(\frac{Z_i'}{\lambda} e^{\pm\lambda u}\right)_t - (Z_i e^{\pm\lambda u})_{\psi} = 0$$

$$\left(\rho u \frac{Z_i'}{\lambda} e^{\pm\lambda u} \pm Z_i e^{\pm\lambda u}\right)_x + \left(\rho \frac{Z_i'}{\lambda} e^{\pm\lambda u}\right)_t = 0$$

The upper sign corresponds to version 1 and the lower sign (plus or minus) corresponds to version 2.

The two additional conservation laws, obtained by adding the divergent equations with subscript i from both version, taken with R_j^- and R_j^+, where $j = 3 - i$, $i = 1, 2$, respectively, can also correspond to the four conservation laws corresponding to the positive constant of separation. As a result we have

$$((R_i^+)_{\psi} R_j^- + (R_i^-)_{\psi} R_j^+)_t - ((R_i^+)_t R_j^- + (R_i^-)_t R_j^+)_{\psi} \equiv$$
$$\equiv \left(-\frac{Z_i'}{\lambda} e^{\lambda u} R_j^- + \frac{Z_i'}{\lambda} e^{-\lambda u} R_j^+\right)_t - (Z_i e^{\lambda u} R_j^- + Z_i e^{-\lambda u} R_j^+)_{\psi} = 0$$

Remarks. As pointed out, the solutions corresponding to the positive constant of separation possess an interesting property. Due to the fact that, in version 1, the expressions for t and ψ contain the same factor $e^{\lambda u}$, we obtain

$$\frac{t}{\psi} = -\frac{Z_i'(p, \lambda)}{Z_i(p, \lambda)}, \quad i = 1, 2$$

after which, we finally have

$$p = F\left(\frac{t}{\psi}, \lambda\right), \quad u = \frac{1}{\lambda} \ln \frac{\psi}{Z_i(p, \lambda)}, \quad i = 1, 2$$

In exactly the same way we have for the solution of version 2

$$p = G\left(\frac{t}{\psi}, \lambda\right), \quad u = -\frac{1}{\lambda} \ln \frac{\psi}{Z_i(p, \lambda)}, \quad i = 1, 2$$

where F and G are certain functions to be determined.

Hence, the solutions corresponding to the positive constant of separation possess the following properties: in the event plane (ψ, t) these solutions are characterized by constant values of the pressure p and a logarithmic relationship between the velocity u and ψ along the ray t/ψ = const.

I wish to thank G. G. Chernyi, Yu. D. Shmyglevskii, S. K. Godunov and G. V. Demidenko for discussing this problem.

This research was supported by the Russian Foundation for Basic Research (01-01-00851).

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Translated by R.C.G.